Stochastic Equations and Processes in physics and biology

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Stochastic differential equations

The Wiener-Khinchin theorem

ACF of a real-valued signal $x(t)$

$$G(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt x(t) x(t+\tau),$$

Forward Fourier transform of $x(t)$

$$\hat{x}(\omega) = \int_0^T dt e^{-i\omega t} x(t).$$

Symmetry of $\hat{x}(\omega)$

$$\hat{x}(-\omega) = \hat{x}(\omega)^*$$

Proof of symmetry

$$\hat{x}(-\omega) = \int_0^T dt e^{i\omega t} x(t) = \left(\int_0^T dt e^{-i\omega t} x(t) \right)^* = \hat{x}(\omega)^*$$

The Wiener-Khinchin theorem

Power spectral density $S(\omega)$ (psd)

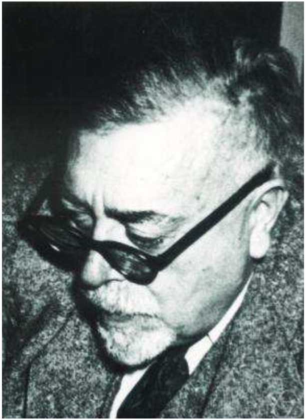
$$S(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \mid \hat{x}(\omega) \mid^2,$$

Wiener-Khinchin theorem

$$S(\omega) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} d\tau e^{-i\omega\tau} G(\tau)$$

Norbert Wiener

American mathematician (1894-1964) proved the theorem in 1930 for a deterministic process $x(t)$



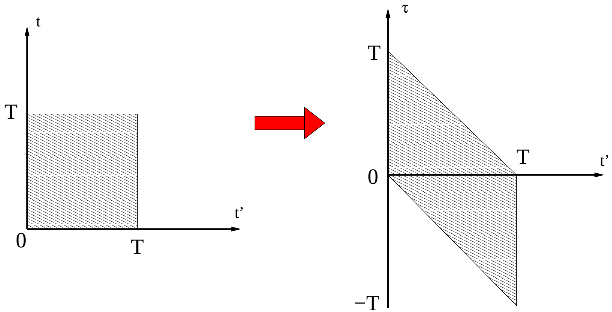
Aleksandr Yakovlevich Khinchin

Russian mathematician (1894-1959) proved the theorem in 1934 for a stochastic process $x(t)$

Wiener-Khinchin theorem

Proof

$$
\begin{aligned}
S(\omega) &= \lim_{T\to\infty} \frac{1}{2\pi T} \int_0^T dt e^{-i\omega t} x(t) \int_0^T dt' e^{i\omega t'} x^*(t') \\
&= \left\{ (t,t') \to (t', \tau = t - t') \right\}
\end{aligned}
$$



Wiener-Khinchin theorem

Proof continued

$$\begin{aligned}
S(\omega) &= \lim_{T\to\infty} \frac{1}{2\pi T}\left[\int_0^T d\tau e^{-i\omega\tau}\int_0^{T-\tau} dt' x(t')x^*(t'+\tau)\right.\\
&+ \left.\int_{-T}^0 d\tau e^{-i\omega\tau}\int_{-\tau}^T dt' x(t')x^*(t'+\tau)\right]\\
&= \frac{1}{2\pi T}\left[\int_{-T}^0 d\tau e^{-i\omega\tau}(TG(\tau)+O(1))\right.\\
&+ \left.\int_0^T d\tau e^{-i\omega\tau}(TG(\tau)+O(1))\right]\\
&= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^T d\tau e^{-i\omega\tau}G(\tau)
\end{aligned}$$

Wiener-Khinchin theorem

Using $G(-\tau) = G(\tau)$

$$\begin{aligned} S(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\tau)e^{\pm i\omega\tau}d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\tau)\cos\omega\tau d\tau \\ &= \frac{1}{\pi}\int_{0}^{\infty} G(\tau)\cos\omega\tau d\tau \end{aligned}$$

psd is an even function

$$S(-\omega) = S(\omega)$$

Wiener-Khinchin theorem

Delta function representation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\omega x} d\omega,$$

ACF and psd

ACF is obtained as inverse Fourier transform of the psd.

$$G(\tau) = \int_{-\infty}^{\infty} e^{\pm i\omega\tau} S(\omega) d\omega.$$

psd is obtained as forward Fourier transform of the ACF.

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\omega\tau} G(\tau) d\tau.$$

Stochastic vs deterministic processes

Deterministic signal $x(t)$

Fourier decomposition

Any deterministic signal $x(t)$ can always be represented as a superposition of periodic functions

$$x(t) = A\cos\Omega t$$

Fourier transform $\hat{x}(\omega)$

$$\begin{aligned}
\hat{x}(\omega) &= \int_0^T dt e^{-i\omega t} A\cos\Omega t = \frac{A}{2}\int_0^T dt e^{-i\omega t}(e^{i\Omega t} + e^{-i\Omega t}) \\
&= \frac{A}{2}\begin{cases} \left(\frac{e^{it(\Omega-\omega)}}{i(\Omega-\omega)} - \frac{e^{-it(\Omega+\omega)}}{i(\Omega+\omega)}\right)\Big|_0^T, & \omega \neq \pm\Omega \\ T, & \omega = \pm\Omega \end{cases} \\
&= T\frac{A}{2}(\delta_{\omega,\Omega} + \delta_{\omega,-\Omega}) + O(T^0)
\end{aligned}$$

deterministic processes: psd

psd $S(\omega)$

$$
\begin{aligned}
S(\omega) &= \lim_{T\to\infty} \frac{\hat{x}(\omega)\hat{x}^*(\omega)}{2\pi T} = \lim_{T\to\infty} \frac{1}{2\pi T}\left(T\frac{A}{2}(\delta_{\Omega,\omega}+\delta_{\omega,-\Omega}) + O(T^0)\right)^2 \\
&= \lim_{T\to\infty} \frac{A^2 T}{8\pi}(\delta_{\Omega,\omega}+\delta_{\omega,-\Omega})^2 + O(T^0) + O(T^{-1}) \\
&= \lim_{T\to\infty} \frac{A^2 T}{8\pi}(\delta^2_{\Omega,\omega}+\delta^2_{\omega,-\Omega}) + O(T^0) + O(T^{-1})
\end{aligned}
$$

Infinite power of a deterministic signal

$\lim_{T\to\infty} S(\omega = \pm\Omega) \to \infty$

deterministic processes: ACF

ACF $G(\tau)$

$$
\begin{aligned}
G(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt x(t)x(t+\tau) = \lim_{T\to\infty} \frac{A^2}{T}\int_0^T dt\ \cos\Omega t \cos\Omega(t+\tau) \\
&= \lim_{T\to\infty} \frac{A^2}{T}\int_0^T dt\ \left(\cos^2\Omega t\cos\Omega\tau - \sin\Omega t\cos\Omega t\sin\Omega\tau\right) \\
&= \lim_{T\to\infty} \frac{A^2}{T}\int_0^T dt\ \left(\frac{1}{2}[1+\cos 2\Omega t]\cos\Omega\tau - \frac{1}{2}\sin 2\Omega t\sin\Omega\tau\right) \\
&= \frac{A^2\cos\Omega\tau}{2} + O(1/T)
\end{aligned}
$$

Infinite correlation time

ACF $G(\tau)$ of a deterministic signal does not vanish as $\tau\to\infty$.

deterministic processes: ACF and psd

Limit of large time window T

As $T \to \infty$, the Wiener-Khinchin theorem implies

$$\begin{aligned} S(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} G(\tau) d\tau = \frac{A^2}{4\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} \cos\Omega\tau d\tau \\ &= \frac{A^2}{8\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\left(e^{i\Omega\tau} + e^{-i\Omega\tau}\right) d\tau \\ &= \frac{A^2}{4}\left(\delta(\omega - \Omega) + \delta(\omega + \Omega)\right) \end{aligned}$$

Only works for square-integrable functions

Stochastic porcess $x(t)$

Special case of a random telegraph process

$$x(t) = \pm 1, \quad \lambda = \mu$$

Recall stationary ACF $G(\tau)$

$$G(\tau) = \frac{(1-(-1))^2\mu^2}{(\mu+\mu)^2}\exp\left[-2\mu|\tau|\right] = e^{-2\mu|\tau|}$$

$G(\tau)$ is square-integrable on $(-\infty, \infty)$

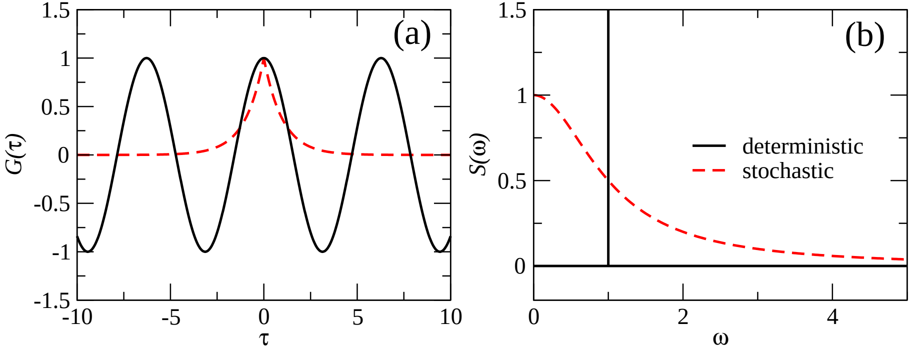
Applying the Wiener-Khinchin theorem

Stochastic porcess $x(t)$

psd from the Wiener-Khinchin theorem

$$
\begin{aligned}
S(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\mu|\tau|}e^{-i\omega\tau}\,d\tau \\
&= \mathrm{Re}\left(2\frac{1}{2\pi}\int_{0}^{\infty} e^{-2\mu\tau}e^{-i\omega\tau}\,d\tau\right) \\
&= \mathrm{Re}\left(\frac{1}{\pi}\frac{e^{-\tau(2\mu+i\omega)}}{-(2\mu+i\omega)}\Big|_0^{\infty}\right) \\
&= \mathrm{Re}\left(\frac{1}{\pi}\frac{1}{(2\mu+i\omega)}\right) \\
&= \frac{2\mu}{\pi(4\mu^2+\omega^2)}
\end{aligned}
$$

Stochastic vs deterministic processes



White noise

Stochastic process $\chi(t)$ with zero mean and zero correlation time

$$\langle \chi(t) \rangle = 0, \qquad \langle \chi(t)\chi(t') \rangle = \delta(t - t')$$

psd

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\tau) e^{i\omega\tau} \, d\tau = \frac{1}{2\pi}$$

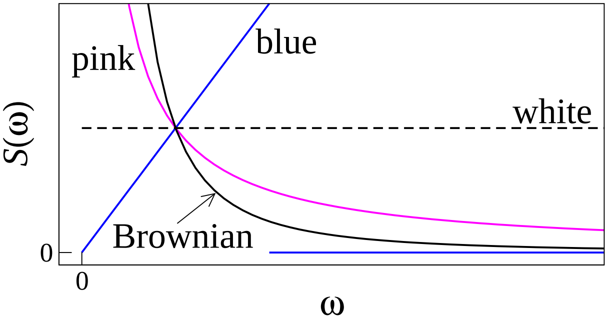
white noise

Equal power density for any frequency

Colored noise

Stochastic process $\eta(t)$ **with any non-constant psd** $S(\omega) \neq constant$

- Pink noise: $S(\omega) = \omega^{-1}$
- Brownian noise (red noise): $S(\omega) = \omega^{-2}$
- Blue noise: $S(\omega) = \begin{cases} \omega, & \omega \in [0, \omega_0] \\ 0, & \text{otherwise} \end{cases}$



Colored noise

Flavours of the noise

For each noise color one can define infinitely many different flavours, depending on the distribution of x at any given time.

Discretized signal

In practice, we descretize $x(t)$ by choosing a time step Δt:

$$x(t_i),\;\; t_i = t_0 + i\Delta t,\;\; (i = 0, 1, 2, 3, \dots)$$

White Gaussian noise

Generated by

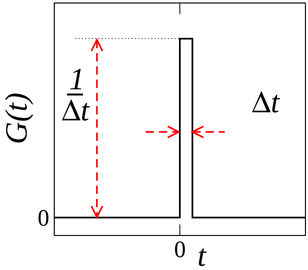
$$x(t_i) \sim \frac{1}{\sqrt{\Delta t}} N(0,1) = N\left(0, \frac{1}{\Delta t}\right)$$

Indeed

$$\langle x(t_i)\rangle = 0, \quad G(t_k - t_i) \quad = \quad \langle x(t_i)x(t_k)\rangle = \begin{cases} 0, & i \neq k \\ \frac{1}{\Delta t}, & i = k \end{cases}$$

Representation of the Dirac delta-function $\delta(t)$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad \int \delta(t)\, dt = 1$$



Uniform white noise

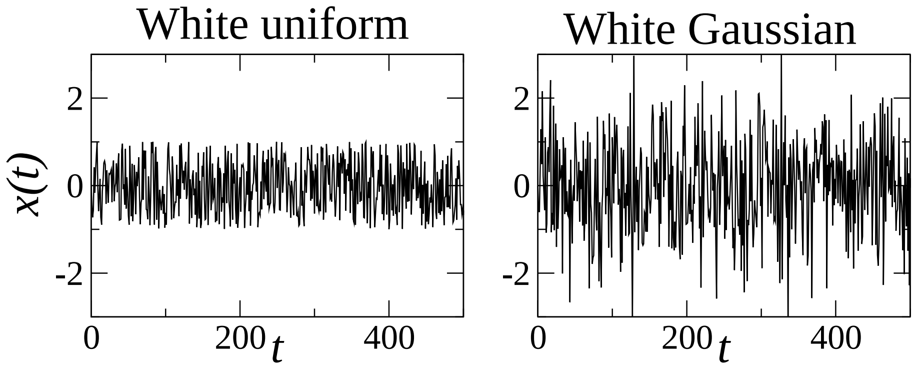
Generated by

$$x(t_i) \sim \sqrt{\frac{3}{\Delta t}} \mathrm{Uniform}([-1, 1])$$

Indeed

$$\begin{aligned} \langle x(t_i) \rangle &= 0 \\ G(t_k - t_i) &= \langle x(t_i) x(t_k) \rangle = \begin{cases} 0, & i \neq k \\ \frac{1}{\Delta t}, & i = k \end{cases} \end{aligned}$$

Gaussian and uniform white noise



The Wiener process

Stochastic process $w(t)$ **with conditional probability**

$$\partial_t p(w,t|w_0,t_0) = \frac{1}{2}\partial_w^2 p(w,t|w_0,t_0).$$

Solution of the diffusion equation with $D = 1/2$

$$p(w,t|w_0,t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}} \exp\left(-\frac{(w-w_0)^2}{2(t-t_0)}\right).$$

Wiener process: independents of increments Δw_i

Markovian property

For any $t_i > t_{i-1}$, the increments $\Delta w_i = w(t_i) - w(t_{i-1})$ are independent with the distribution

$$p(\Delta w_i) = p(w_i, t_i | w_{i-1}, t_{i-1}) = \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\frac{(w_i - w_{i-1})^2}{2(t_i - t_{i-1})}\right)$$

$$\begin{aligned} \Delta w_i &\sim N(0, (t_i - t_{i-1})) = N(0, \Delta t) \\ \langle (\Delta w_i)^2 \rangle &= \Delta t = t_i - t_{i-1} \end{aligned}$$

Wiener process as a solution of stochastic differential equation

Derivative of the Wiener process

White Gaussian noise $\xi(t)$

$$\xi(t) = \frac{\Delta w}{\Delta t} \sim N\left(0, \frac{1}{\Delta t}\right) = \frac{1}{\sqrt{\Delta t}} N(0,1)$$

Stochastic differential equation for $w(t)$

$$\frac{dw(t)}{dt} = \dot{w}(t) = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \xi(t)$$

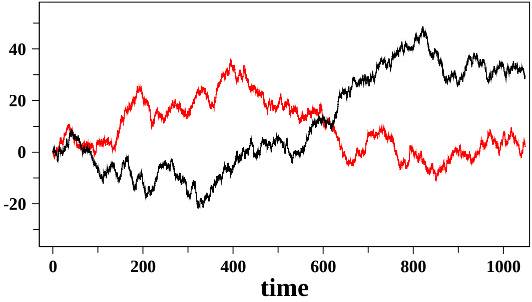
Wiener process and Brownian motion

Brownian motion

The Wiener process $w(t)$ is generated by

$$w(t_{i+1}) = w(t_i) + N(0,1)\sqrt{\Delta t}, \quad (i = 0, 1, 2, 3, \dots).$$

Describes the coordinate of a Brownian particle with the diffusion coefficient $D = 1/2$.



Computation of Power spectral density

Ensemble average of the square of Fourier transformed $x(t)$

$$\langle \hat{x}(\omega)\hat{x}^*(\omega')\rangle = \int dt\, dt' e^{-i\omega t} e^{i\omega' t'} \langle x(t)x(t')\rangle,$$

For a stationary process $x(t)$

$$\langle x(t)x(t')\rangle = G(t-t'),$$

Connection with $S(\omega)$

$$\begin{aligned}
\langle \hat{x}(\omega)\hat{x}^*(\omega')\rangle &= \int dt\, dt' e^{-i\omega t} e^{i\omega' t'} G(t-t') = \left\{(t,t') \to (t', \tau = t-t')\right\} \\
&= \int dt' e^{i(\omega'-\omega)t'} \int d\tau e^{-i\omega' \tau} G(\tau) \\
&= 2\pi \int dt' e^{i(\omega'-\omega)t'} S(\omega') = (2\pi)^2 S(\omega')\delta(\omega-\omega').
\end{aligned}$$

psd of the Wiener process $dW/dt = \xi(t)$

In the Fourier space

$$i\omega\hat{x}(\omega) = \hat{\xi}(\omega),$$

Consequently

$$\langle \hat{x}(\omega)\hat{x}^*(\omega')\rangle = \frac{\langle \hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle}{(i\omega)(-i\omega')}$$

By the definition of the white noise

$$\langle \hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle = (2\pi)^2\left(\frac{1}{2\pi}\int G_\xi(\tau)d\tau e^{-i\omega\tau}\right)\delta(\omega-\omega') = (2\pi)\delta(\omega-\omega'),$$

$$\begin{aligned}\langle \hat{x}(\omega)\hat{x}^*(\omega')\rangle &= \frac{2\pi\delta(\omega-\omega')}{\omega^2} = (2\pi)^2 S(\omega)\delta(\omega-\omega')\\ S(\omega) &= \left(\frac{1}{2\pi}\right)\frac{1}{\omega^2}.\end{aligned}$$

ACF of the Wiener process

Using the Wiener-Khinchin theorem

$$G(\tau) = \int_{-\infty}^{\infty} \frac{1}{2\pi\omega^2} e^{i\omega\tau} d\omega \sim \tau.$$

From ACF to psd

The backward Fourier transformation

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau e^{-i\omega\tau}\, d\tau$$

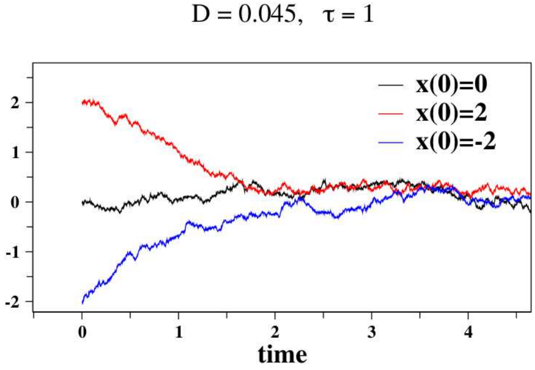
does not exist, as the function $G(\tau) \sim \tau$ is not a square-integrable on the interval $(-\infty, \infty)$. Physically, it means that the Wiener process is not a stationary process.

Ornstein-Uhlenbeck process

Stochastic ODE for $x(t)$

$$\dot{x}(t) = -\frac{x}{\tau} + \frac{1}{\tau}\sqrt{2D}\xi(t),$$

with $\xi(t)\ldots$ white noise
$D\ldots$ strength parameter
$\tau\ldots$ correlation time

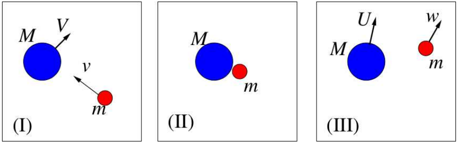


Ornstein-Uhlenbeck process

Massive particle of mass M in a gas of molecules m

Momentum and energy conservation in a single elastic collision

$$
\begin{aligned}
M\frac{V^2}{2} + m\frac{v^2}{2} &= M\frac{U^2}{2} + m\frac{w^2}{2} \\
M\vec{V} + m\vec{v} &= M\vec{U} + m\vec{w}
\end{aligned}
$$



Ornstein-Uhlenbeck process

Change of the momentum of the massive particle

$$\Delta\vec{P} = M\vec{U} - M\vec{V} = -\frac{2m}{M+m}(M\vec{V}) + \frac{2M}{M+m}(m\vec{v})$$

Number of collisons within time interval δt

$$\delta t \ll 1, \quad \left\|\frac{\delta P(t)}{P(t)}\right\| \ll 1, \quad I_i(t,\delta t) = (1,0)$$

Total momentum change

$$\Delta\vec{P} = -\left(\sum_{i=1}^{N} \frac{2m}{M+m} I_i(t,\delta t)\right)\vec{P} + \left(\sum_{i=1}^{N} \frac{2M}{M+m} I_i(t,\delta t)\vec{p}_i\right)$$

Langevin equation

Large number of uncorrelated collision events within dt

$$\frac{d\vec{P}}{dt} = -\alpha(P)\vec{P} + \vec{\xi}(t)$$

Paul Langevin (1872-1946): French physicist

psd of the Ornstein-Uhlenbeck process

In the Fourier space

$$i\omega\hat{x}(\omega) = -\frac{\hat{x}(\omega)}{\tau} + \frac{\sqrt{2D}}{\tau}\hat{\xi}(\omega)$$

Solving for $\hat{x}(\omega)$

$$\hat{x}(\omega) = \frac{\sqrt{2D}}{\tau}\frac{\hat{\xi}(\omega)}{i\omega + 1/\tau}$$

Taking ensemble average

$$\langle\hat{x}(\omega)\hat{x}^*(\omega')\rangle = 4\pi D\frac{\delta(\omega-\omega')}{1+\tau^2\omega^2}.$$

Taking ensemble average

$$S(\omega) = \frac{D}{\pi}\frac{1}{1+\tau^2\omega^2}.$$

ACF of the Ornstein-Uhlenbeck process

Relation to random telegraph process

Note that the functional form of $S(\omega)$ is identical with the psd of a random telegraph process. Compare

$$S(\omega) = \frac{2\mu}{\pi(4\mu^2 + \omega^2)} \quad vs \quad S(\omega) = \frac{D}{\pi}\frac{1}{1 + \tau^2\omega^2}$$

This implies that the ACF of the Orstein-Uhlenbeck process has the form

$$G(t) = \frac{D}{\tau}e^{-|t|/\tau}$$

ACF of the Ornstein-Uhlenbeck process

Check using the Wiener-Khinchin theorem

$$\begin{aligned} G(t) &= \int_{-\infty}^{\infty} \frac{D}{\pi} \frac{e^{i\omega t}}{1+\tau^2\omega^2}\, d\omega \\ &= \text{residual} \left(\frac{De^{i\omega t}}{\pi\tau^2(\omega - i/\tau)(\omega + i/\tau)} \right) \Big|_{\omega = i\tau^{-1}} \\ &= \frac{D}{\tau} e^{-t/\tau}, \ \ (t \geq 0) \end{aligned}$$

Correlation time

ACF decay with the characteristic rate of τ.

2D plasma in magnetic field

Example

A charged particle, confined to move on a plane in a constant magnetic field B, which is perpendicular to the plane of motion.

$$\dot{\boldsymbol{r}} \;=\; \boldsymbol{v},\;\; \dot{\boldsymbol{v}} = \frac{q}{m}\boldsymbol{v}\times\boldsymbol{B} - \gamma\boldsymbol{v} + \sqrt{2\gamma\frac{kT}{m}}\boldsymbol{\xi}(t),$$

with $\gamma \ldots$ the damping coefficient
$T \ldots$ the absolute temperature
$m \ldots$ mass of the particle
$q \ldots$ charge of the particle
$k \ldots$ Boltzmann constant
Question: Determine the power spectrum of the velocity components v_x and v_y.

2D plasma in magnetic field