Stochastic Equations and Processes in Physics and Biology

Andrey Pototsky

Swinburne University

AMSI 2017

Probability: basic concepts and definitions

Discrete case

$X \ldots$ a discrete random variable
$P(X) \ldots$ *probability distribution function*

$$P(X_i) = \lim_{n\to\infty} \frac{1}{n} \#\{X = X_i\}, \quad P(X_i) \in [0,1]$$

with
$\#\{X = X_i\} \ldots$ number of outcomes with $X = X_i$.
Normalization condition

$$\sum_{X_i \in \Omega} P(X_i) = 1,$$

$\Omega \ldots$ set of all possible values of X

Continuous case

$X\dots$ continuous random variable from $X \in [a, b]$
$x\dots$ specific value of X
$\rho(x)\dots$ *probability density function (pdf)*

$$\rho(x) = \lim_{n\to\infty,\ dx\to 0} \frac{\#\{X \in [x, x+dx]\}}{n\,dx}.$$

Probability to find X in a narrow interval $[x, x+\delta x]$

$$\Pr(X \in [x, x+\delta x]) = \rho(x)\,\delta x.$$

Normalization condition

$$\int_a^b \rho(x)\,dx = 1.$$

cumulative distribution function (cdf): $F(x)$

$$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} \rho(x)\, dx.$$

This shows that

$$\rho(x) = \frac{dF(x)}{dx}.$$

Note that $F(x)$ and $\rho(x)$ are defined on the whole line $x \in (-\infty, +\infty)$.

$$\text{If} \ \ x \in [a, b]$$

$$\rho(x) = 0, \qquad x \notin [a, b]$$

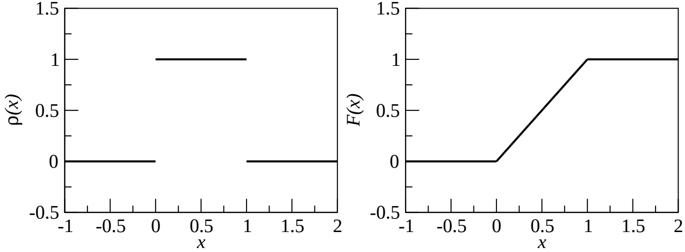
$F(x)$ **is monotonically increasing with**

$$F(-\infty) = 0, \quad \text{and} \quad F(+\infty) = 1$$

Uniform distribution

$X \sim$ **uniformly distributed on** $[0,1]$

$$\text{pdf}: \ \rho(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \qquad \text{cdf}: \ F(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \in [0,1] \\ 1, & x \geq 1 \end{cases}$$



Expected value (average or mean)

In theory for a discrete rv X

$$E(X) = \langle X \rangle = \sum_{X_i \in \Omega} P(X_i) X_i$$

In theory for a continuous rv X

$$E(X) = \langle X \rangle = \int_{-\infty}^{+\infty} x \rho(x)\, dx$$

For any given function $Y = f(X)$

$$E(Y) = \langle Y \rangle \quad = \quad \sum_{X_i \in \Omega} f(X_i) P(X_i) \Rightarrow \text{discrete},$$

$$E(Y) = \langle Y \rangle \quad = \quad \int_{-\infty}^{+\infty} f(x) \rho(x)\, dx \Rightarrow \text{continuous}$$

Expected value in an experiment

In a random experiment with n tries

$$\langle X \rangle = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Unbiased estimator for the mean

$X_i \ldots$ independent and identically distributed (iid) rvs with

$$E(X_i) = \mu$$

Show that

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mu$$

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n}{n}\mu = \mu$$

Variance and standard deviation

In theory for a discrete rv X

$$\mathrm{Var}(X) = \sum_{X_i \in \Omega} P(X_i)(X_i - E(X))^2$$

In theory for a continuous rv X

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 \rho(x)\, dx = \int_{-\infty}^{+\infty} x^2 \rho(x)\, dx - (\langle X \rangle)^2$$

Standard deviation σ

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

Proof of $\mathrm{Var}(X) = E(X^2) - E(X)^2$

$$
\begin{aligned}
&\mathrm{Var}(X) \\
= &\int_{-\infty}^{+\infty} (x - E(X))^2 \rho(x)\, dx \\
= &\int_{-\infty}^{+\infty} [x^2 + E(X)^2 - 2xE(X)]\rho(x)\, dx \\
= &\int_{-\infty}^{+\infty} x^2 \rho(x)\, dx + E(X)^2 \int_{-\infty}^{+\infty} \rho(x)\, dx - 2E(X) \int_{-\infty}^{+\infty} x\, \rho(x)\, dx \\
= &\int_{-\infty}^{+\infty} x^2 \rho(x)\, dx + E(X)^2 - 2E(X)^2 = E(X^2) - E(X)^2
\end{aligned}
$$

Relation between $E(X)$ and $E(X^2)$:

$$\langle X^2 \rangle \geq \langle X \rangle^2$$

Expected value (average or mean)

For a uniform distribution

$$E(X) = \int_0^1 x\,dx = \frac{1}{2}$$

$$\mathrm{Var}(X) = \int_0^1 x^2\,dx - (E(X))^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\text{standard deviation} = \sigma = \sqrt{\mathrm{Var}(X)} = \frac{1}{\sqrt{12}}$$

$$E(\sin(X)) = \int_0^1 \sin(x)\,dx = -\cos(x)\Big|_0^1 = 1 - \cos(1)$$

Variance and standard deviation in an experiment

In a random experiment with n tries

$$\mathrm{Var}(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \langle X\rangle)^2, \quad \text{with} \quad \langle X\rangle = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Example

Let X_i be iid with $E(X_i) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Show that

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \langle X\rangle)^2$$

is an unbiased estimator for σ^2.

For this, show that

$$E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \langle X\rangle)^2\right) = \sigma^2$$

Normal distribution

The *normal* rv X is a continuous rv with pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad (-\infty < x < \infty)$$

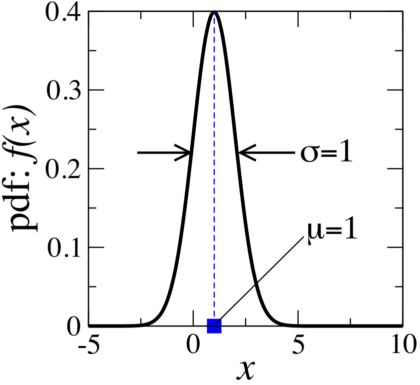
$$X \sim N(\mu, \sigma^2), \quad E(X) = \mu, \quad \mathrm{Var}(X) = \sigma^2$$

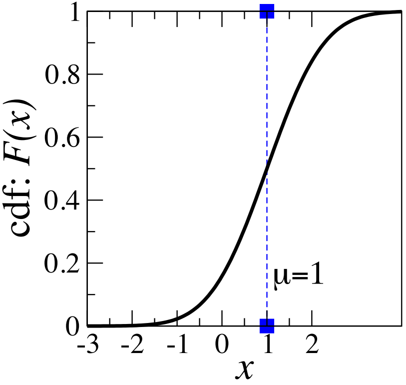
Basic integrals to solve

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx &= 1 \\
\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx &= \mu \\
\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx &= \sigma^2 + \mu^2
\end{aligned}$$

Normal distribution

Graph of the pdf $f(x)$ and the cdf $F(x)$ for a normal rv $X \sim N(1,1)$.





Z-score and standard normal distribution

$$X \sim N(\mu, \sigma^2)$$

then

$$Z = \frac{X - \mu}{\sigma}$$

follows the standard normal distribution, i.e. $Z \sim N(0, 1)$

$$\text{pdf}(z) = f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad (-\infty < z < \infty)$$

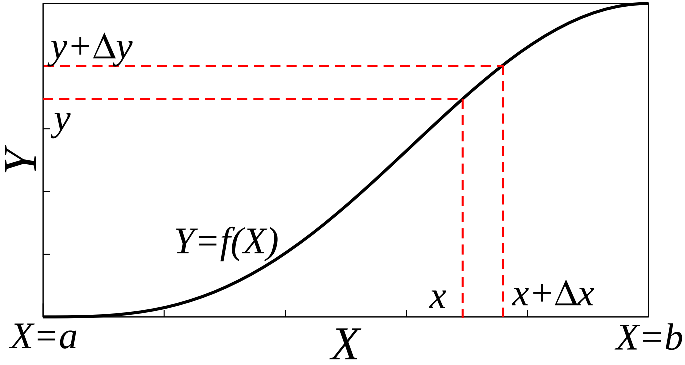
cdf: $F(z)$

$$F(z) = \int_{-\infty}^{z} \text{pdf}(z)\, dz = \frac{1}{2}\left(\text{erf}(z/\sqrt{2}) + 1\right)$$

Error function $\text{erf}(z) = 2/\sqrt{\pi} \int_0^z e^{-x^2}\, dx$ **is tabulated**

Change of variables (continuous case): Given a pdf for X, what is the pdf for $Y = f(X)$?

$\rho(x)\ldots$ pdf of X on $x \in [a, b]$
$Y = f(X)\ldots$ one-to-one function on $[a, b]$
$p(y)\ldots$ pdf of Y on $y \in [f(a), f(b)]$ to be found



Change of variables (continuous case)

Probability for Y to be in $[y, y + \Delta y]$

$$\Pr(Y \in [y, y + \Delta y]) = p(y)\,\Delta y$$

Probability for X to be in $[x, x + \Delta x]$

$$\Pr(X \in [x, x + \Delta x]) = \rho(x)\,\Delta x$$

These probabilities are identical

$$p(y)\,\Delta y = \rho(x)\,\Delta x, \Rightarrow \qquad p(y) = \rho(x)\frac{\Delta x}{\Delta y}$$

$$p(y) = \rho(x)\frac{1}{(\Delta y / \Delta x)} = \rho(x)\frac{1}{f'(x)}$$

Using $x = f^{-1}(y)$, we find

$$p(y) = \rho(f^{-1}(y))\,\frac{1}{f'(f^{-1}(y))}.$$

Change of variables: example

Example

Generate random numbers $y \geq 0$, with a given pdf $p(y)$, using a uniformly distributed random numbers x from $x \in [0, 1]$.

Change of variables: solution

Looking for $y = f(x)$ such that

$$p(y) = \frac{\rho(x)}{f'(x)} = \frac{1}{f'(x)}, \text{ with } \rho(x) = 1$$

with $f'(x) = dy/dx$, we find

$$p(y)\,dy = dx$$

integrating

$$\int_0^y p(y)\,dy = \int_0^x dx = x,$$

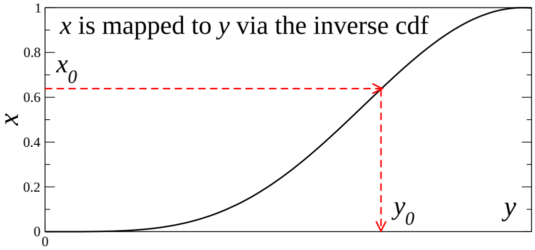
Recalling the definition of the cdf $G(y)$ of y

$$G(y) = \int_{-\infty}^y p(y)\,dy = \int_0^y p(y)\,dy$$

solution: continued

Transformation formulae

$$x = G(y) \ \Rightarrow \ y = G^{-1}(x)$$



For exponential distribution: $p(y) = \alpha \exp(-\alpha y)$, $y \geq 0$

$$G(y) = \int_0^y \alpha \exp(-\alpha y)\, dy = 1 - \exp(-\alpha y)$$

inverting the cdf

$$y = -\alpha^{-1} \ln(1 - x)$$

Generating a Gaussian rv

cdf of $Z \sim N(0,1)$

$$G(z) = (1/2)\left(\mathrm{erf}(z/\sqrt{2}) + 1\right) = X, \qquad X \text{ uniform on } [0,1]$$

Solving for z

$$z = \sqrt{2}\,\mathrm{erf}^{-1}(2X-1)$$

Inverse error function method

Computationally slow, as one needs to evaluate $\mathrm{erf}^{-1}(\dots)$

Box-Muller algorithm

Let U_1 and U_2 are independent and uniform on $(0,1)$. Then

$$Z_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2), \text{ and } Z_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

are independent with standard normal distribution $(Z_1, Z_2) \sim N(0,1)$.

Probability and events

mutually disjoint events A and B are such that

$$A \cap B = \emptyset,$$

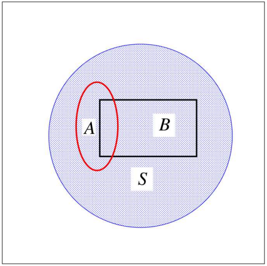
where $\emptyset$ denotes an empty set.

If A_i, $i = 1, 2, 3, 4, \ldots$ are mutually disjoint, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

For any A and B, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Probability and events

Conditional probability $\Pr(A|B)$ is defined for any two events A and B as the probability of the event A given that the event B has certainly occurred.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Example: roll of a die

- a)

$$A = \{1, 2, 3\}, \;\; B = \{1, 2, 5, 6\}, \;\; P(A|B) = \frac{P(\{1, 2\})}{P(\{1, 2, 5, 6\})} = \frac{2/6}{4/6} = 0.5$$

- b)

$$A = \{1, 2\}, \quad B = \{5, 6, 4, 3\}, \quad P(A|B) = \frac{P(\emptyset)}{P(\{5, 6, 4, 3\})} = 0$$

Probability

Example

Your neighbor has two children. You know that the name of one of them is John. What is the probability that your neighbor has two boys?

Solution: construct a table with all possible outcomes

Event	first child	second child	probability
(b, b)	boy	boy	$1/4$
(b, g)	boy	girl	$1/4$
(g, b)	girl	boy	$1/4$
(g, g)	girl	girl	$1/4$

Denote $A = (\text{two boys}) = P(b, b)$ and
$B = (\text{one is a boy}) = P(\{(b, b), (b, g), (g, b)\})$.
Then $A \cap B = A$. Consequently

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Independent events

Joint distribution

Any two events A and B are *independent* if the joint distribution can be factorized

$$P(A \cap B) = P(A)P(B)$$

As a consequence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B).$$

Sum of two normal rv $Y = X_1 + X_2$

Example

Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ find $Y \sim N(E(Y) =?, \mathrm{Var}(Y) =?)$

$$E(Y) = E\left(X_1 + X_2\right) = E(X_1) + E(X_2) = \mu_1 + \mu_2$$

$$\begin{aligned}
\mathrm{Var}(Y) &= E((X_1 + X_2)^2) - (\mu_1 + \mu_2)^2 \\
&= E(X_1^2 + X_2^2 + 2X_1X_2) - (\mu_1 + \mu_2)^2 \\
&= E(X_1^2) + E(X_2^2) + 2E(X_1X_2) - (\mu_1 + \mu_2)^2 \\
&= \sigma_1^2 + \mu_1^2 + \sigma_2^2 + \mu_2^2 + 2\mu_1\mu_2 - (\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2) \\
&= \sigma_1^2 + \sigma_2^2
\end{aligned}$$

Only works if X_1 and X_2 are independent

$$E(X_1X_2) = E(X_1)E(X_2)$$

Ideal gas of active particles

Example

A particle moves with a constant absolute velocity V in a direction that changes randomly in time. For a gas of such active particles with a given concentration ρ_0, the distribution of the direction of motion is uniform.

- Find the pressure in the gas
- Determine the distribution of the relative velocity $U = |\boldsymbol{u}_1 - \boldsymbol{u}_2|$

1D case:

$$u_x = \pm V$$

Ideal gas of active particles: 1D

Number of hits dN per unit area in time dt

$$dN = \frac{1}{2}\rho_0 V dt$$

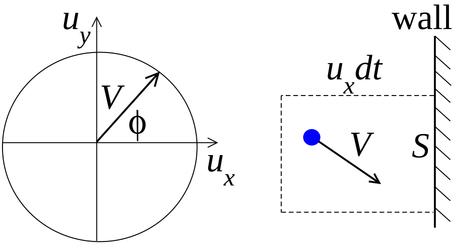
Pressure P

$$P = \frac{m\Delta V}{dt} dN = \frac{\alpha}{2} m\rho_0 V^2, \quad \alpha = \begin{cases} 2 & \text{elastic} \\ 1 & \text{inelastic} \end{cases}$$

Distribution of the relative velocity $U = |\boldsymbol{u}_1 - \boldsymbol{u}_2|$

$$\Pr(U = 0) = \frac{1}{2}, \qquad \Pr(U = 2V) = \frac{1}{2}$$

Ideal gas of active particles: 2D



Number of hits dN per unit area in time dt

$$dN = \rho_0 dt \int_0^V f(u_x) u_x \, du_x, \quad f(u_x) \ldots \text{pdf of } u_x$$

Associated problem:

Find the distribution of the projection of the velocity onto any given direction

Ideal gas of active particles: 2D

Projection onto x axis

$$u_x = V \cos\phi$$

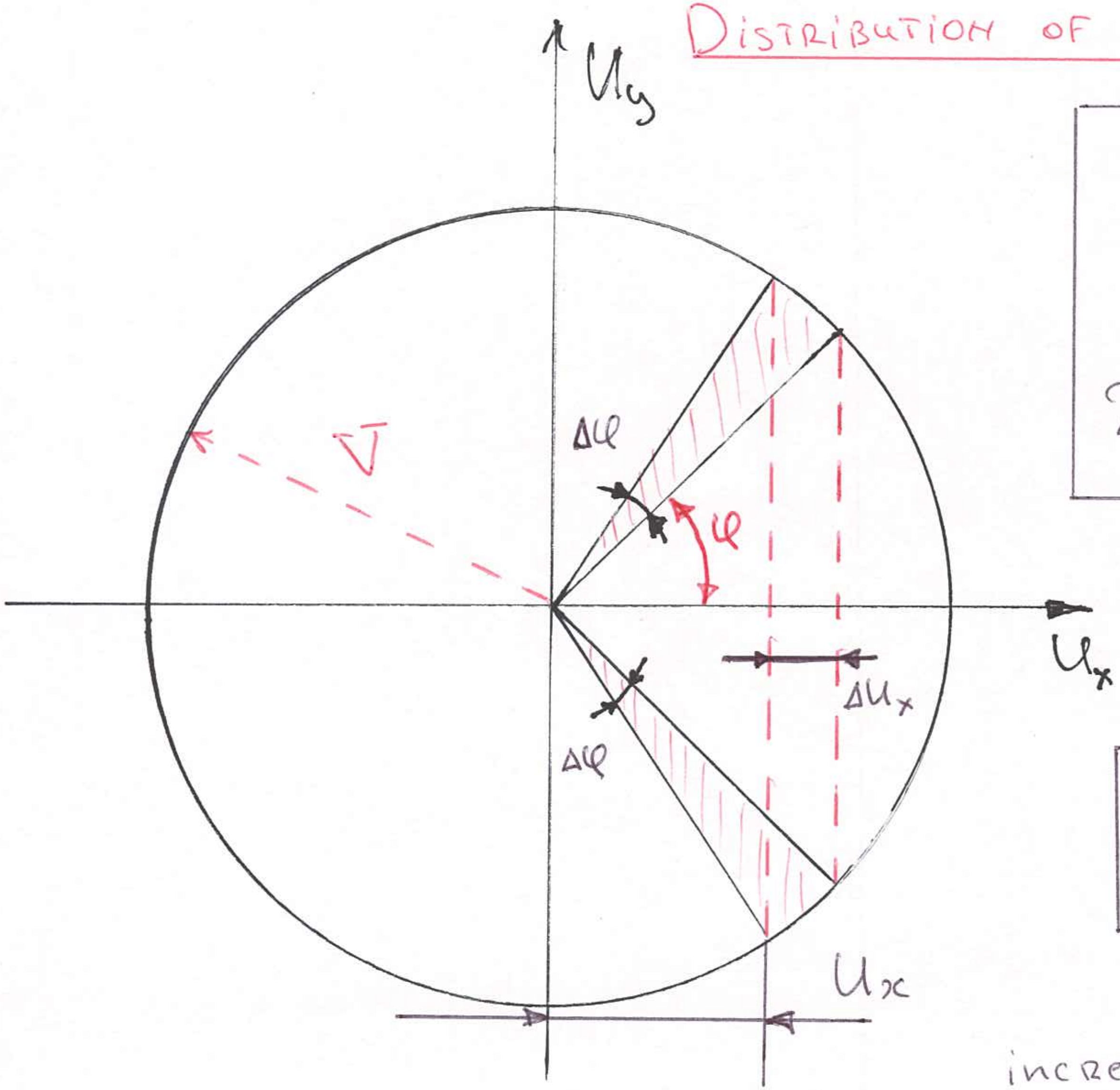
Distribution of the angle ϕ is uniform on $[0, 2\pi]$

$$\mathrm{pdf}(\phi) = \frac{1}{2\pi}$$

Changing variables: $\phi \Rightarrow u_x$

Note that the mapping $u_x = V\cos\phi$ is not one-to-one!

Distribution of the projection onto „x"-axis: 2D



$$P_R\left(U_x \in [U_x, U_x + \Delta U_x]\right) = 2 \cdot P_R\left(\varphi \in [\varphi, \varphi + \Delta\varphi]\right)$$

$$\Delta\varphi \ll \varphi$$

$$\Delta U_x \ll U_x$$

$$-f(U_x)\,\Delta U_x = 2 \cdot \frac{1}{2\pi}\,\Delta\varphi$$

„minus" because: increasing φ => decreasing U_x

$$f(U_x) = \frac{1}{\pi}\left|\frac{\Delta\varphi}{\Delta U_x}\right| \Rightarrow$$

$$f(U_x) = \frac{1}{\pi}\frac{1}{V\sin\varphi} = \frac{1}{\pi V\sqrt{1-\cos^2\varphi}} = \frac{1}{\pi\sqrt{V^2 - U_x^2}}$$

Ideal gas of active particles: 2D

pdf of u_x

$$f(u_x)\,du_x = -2\frac{1}{2\pi}\,d\phi \Rightarrow f(u_x) = \frac{1}{\pi}\left|\frac{1}{du_x/d\phi}\right|$$

Using

$$u_x = V\cos\phi, \qquad \frac{du_x}{d\phi} = -V\sin\phi = -V\sqrt{1-\cos^2\phi}$$

we obtain

$$f(u_x) = \frac{1}{\pi V\sqrt{1-\cos^2\phi}} = \frac{1}{\pi\sqrt{V^2-u_x^2}}.$$

Number of hits per unit area in time dt

$$\frac{dN}{dt} = \rho_0\int_0^V \frac{u_x du_x}{\pi\sqrt{V^2-u_x^2}} = \rho_0\frac{V}{\pi}$$

Ideal gas of active particles: 2D

Pressure

$$P = \rho_0 \int_0^V du_x \, \frac{2mu_x^2}{\pi\sqrt{V^2 - u_x^2}} = \frac{\rho_0 m V^2}{2} = \frac{\rho_v V^2}{2}$$

Ideal gas of active particles: 2D

Relative velocity

$$U = |\boldsymbol{u}_1 - \boldsymbol{u}_2|$$

Associated problem:

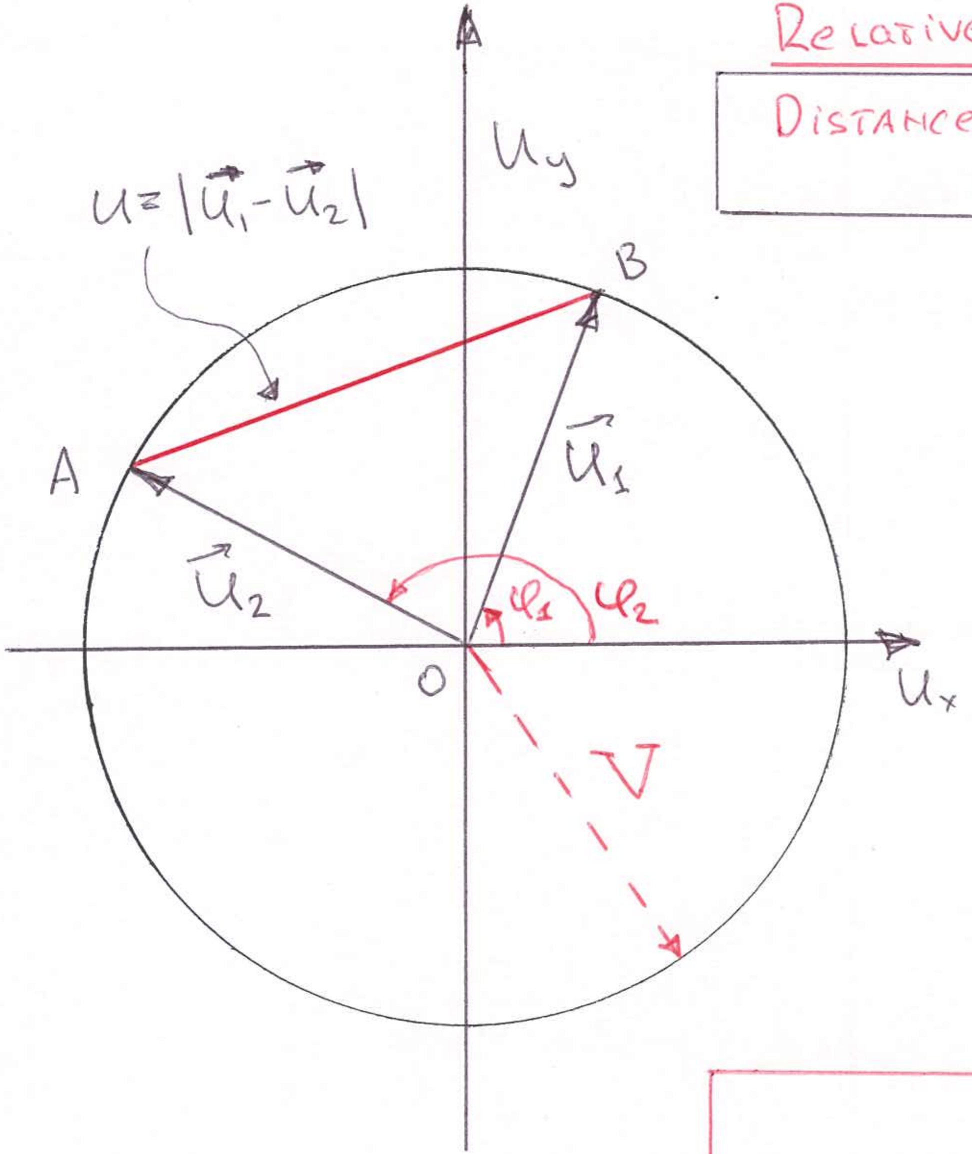
Distribution of the distance between two random points on a circle

Ideal gas of active particles: 2D

Distribution of $\Psi = \phi_2 - \phi_1$

Because ϕ_1 and ϕ_2 are independent, we can fix one angle at an arbitrary value, e.g. ϕ_1-fixed, and look at the distribution of ϕ_2.

- Because ϕ_1 and ϕ_2 are uniform on $[0, 2\pi]$
- $\Psi = \phi_2 - \phi_1$ is also uniform on $[0, 2\pi]$



Relative velocity in 2D <=>

Distance between two random points on a circle

From ΔAOB:

$$U = 2V \sin\left|\frac{\varphi_2 - \varphi_1}{2}\right|$$

For fixed φ_1:

$$Pr\left(\varphi_2 \in [\varphi_1, \varphi_1 + \Delta]\right) =$$

$$cdf(\varphi_1 + \Delta) - cdf(\varphi_1) = \frac{\varphi_1 + \Delta}{2\pi} - \frac{\varphi_1}{2\pi} =$$

$$= \frac{\Delta}{2\pi}$$

$\varphi_2 - \varphi_1$ is uniform on $[0, 2\pi]$

Ideal gas of active particles: 2D

$$\begin{aligned}
\mathrm{cdf}(\Delta) &= \Pr(0 \le \Psi = \phi_2 - \phi_1 \le \Delta) \\
&= \int_0^{2\pi} d\phi_1 \ \Pr(\phi_2 \in [\phi_1, \phi_1 + \Delta] | \phi_1) \times \mathrm{pdf}(\phi_1) \\
&= \int_0^{2\pi} d\phi_1 \ \Pr(\phi_2 \in [\phi_1, \phi_1 + \Delta]) \times \mathrm{pdf}(\phi_1) \\
&= \int_0^{2\pi} d\phi_1 \ [\mathrm{cdf}(\phi_1 + \Delta) - \mathrm{cdf}(\phi_1)] \times \mathrm{pdf}(\phi_1) \\
&= \int_0^{2\pi} d\phi_1 \left[\frac{\phi_1 + \Delta}{2\pi} - \frac{\phi_1}{2\pi}\right] \times \frac{1}{2\pi} = \frac{\Delta}{2\pi}
\end{aligned}$$

Ψ **is uniform on** $[0, 2\pi]$

$$\mathrm{cdf}(\Psi) = \frac{\Psi}{2\pi} \Rightarrow \mathrm{pdf}(\Psi) = \frac{1}{2\pi}$$

Ideal gas of active particles: 2D

periodicity of angles

$\phi_2 - \phi_1$ is uniform only for periodic boundary conditions.

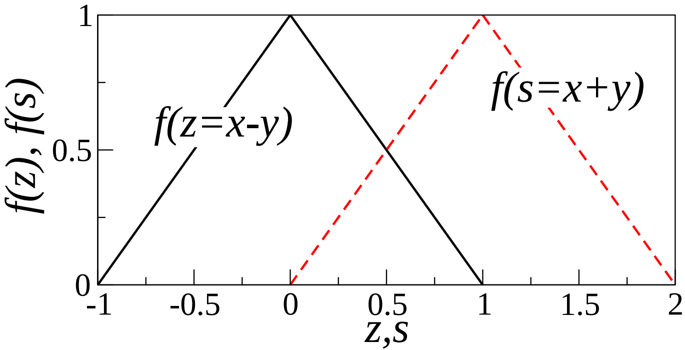
Convolution of probability distributions

If x and y are not periodic and independent on $[0, a]$, then $z = x - y$ and $s = x + y$ are distributed according to:

- $$f(z) = \int_0^a \mathrm{pdf}(x)\mathrm{pdf}(z + x)\, dx$$
- $$f(s) = \int_0^a \mathrm{pdf}(x)\mathrm{pdf}(s - x)\, dx$$

Sum of independent random variables

Sum (difference) of two uniform rvs $(x, y \in [0,1])$



$$f(z) = \begin{cases} z+1, & z \in [-1,0] \\ 1-z, & z \in [0,1] \end{cases} \qquad f(s) = \begin{cases} z, & z \in [0,1] \\ 2-z, & z \in [1,2] \end{cases}$$

Sum of two normal rvs

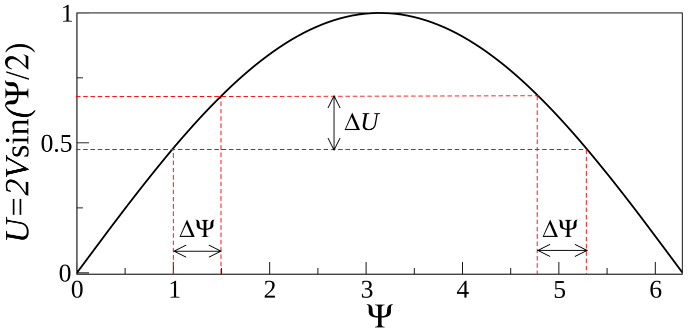
Example

Using the convolution formulae, show that the sum of two independent normal variables $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ is normally distributed with $E(X_1 + X_2) = \mu_1 + \mu_2$ and $\mathrm{Var}(X_1 + X_2) = \sigma_1^2 + \sigma_2^2$.

Ideal gas of active particles: 2D

Distribution of $U = 2V\sin(\Psi/2)$, with Ψ uniform on $[0, 2\pi]$.

$U = 2V\sin(\Psi/2)$ is not a one-to-one function on $[0, 2\pi]$



Ideal gas of active particles: 2D

$$\mathrm{pdf}(U)\Delta U = 2\frac{1}{2\pi}\Delta\Psi \Rightarrow \mathrm{pdf}(U) = \frac{1}{\pi}\left|\frac{1}{dU/d\Psi}\right|$$

$$\mathrm{pdf}(U) = \frac{1}{\pi}\frac{1}{V\cos\left(\Psi/2\right)} = \frac{1}{\pi V\sqrt{1-\sin^2\left(\Psi/2\right)}}$$

$$\mathrm{pdf}(U) = \frac{2}{\pi\sqrt{(2V)^2-U^2}}$$

Average relative velocity

$$\langle U\rangle = \int_0^{2V}\frac{2U\,dU}{\pi\sqrt{(2V)^2-U^2}} = \frac{4V}{\pi}$$

Exercises

- Determine the pressure in the ideal gas of active particles in 3D
- Determine the relative velocity of the active particles in 3D
- Derive the equation of state of an ideal gas, with the Maxwell distribution of the velocities

$$p(\boldsymbol{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\boldsymbol{v}^2}{2kT}\right)$$